Self-Erecting Inverted Pendulum: Swing up and Stabilization Control

S. McGilvray

(Winner of the IEEE Life Member Award for best paper from the Central Canada Council for the IEEE Student Paper Contest, 2002)
Contents

Abstract 3

1. Introduction 3

2. Modelization 4
   2.1 Inverted Pendulum System 4
   2.2 Nonlinear Dynamical Model 5
   2.3 Linearization of the Dynamical Equations 6
   2.4 Control Force / Electro-Mechanical System Equations 7
   2.5 State Space Representation 8

3. Control Design 9
   3.1 Open Loop Swing up Control Design 9
   3.2 System Controllability 10
   3.3 Closed Loop Stabilization Control Design 11

4. Experimental Setup 14
   4.1 Mechanical System 14
   4.2 Motor and Power Module Driver 14
   4.3 Position Sensing Electronics 15
   4.4 Computer Interface Hardware 16

5. Software Interface 17

6. Simulations and Experimental Results 20
   6.1 Numerical Control Design Example 20
   6.2 Simulation Results 21
   6.3 Experimental Results 24

7. Conclusions 26

Acknowledgements 29

References 29
Abstract

Inverted pendulum control is a well-known and challenging problem, which is generally associated to attitude control of a rocket during take off. This system consists of one input generating the cart driving force, and two outputs (linear position of the cart and angular position of the rod). The objective of this experimental design is to swing up the pendulum rod using an appropriate open loop control. Once the angular position has reached a specified capture range a closed loop control law will stabilize the pendulum at its unstable equilibrium point while maintaining a specified linear set-point.

1 Introduction

The goals of the self-erecting inverted pendulum project were to design and build an experimental model of the system, simulate and implement an open loop controller for swing up, and design a Linear Quadratic Regulator (LQR) state feedback controller for stabilization. Finally, it was important to design a user-friendly graphical interface using Visual Basic.

The problem of stabilizing the inverted pendulum is very challenging, and requires a system designed with precision and excellent position sensors. Any measurement or manufacturing errors can result in increased friction on both the linear motion of the cart and the angular motion of the rod. Since the friction of the system is not taken into account for the modelization or in the control design it is important to have as little friction present as possible.

The objectives of the control design are to swing the pendulum upright using an open loop control function, then stabilize the pendulum at its unstable point of equilibrium while maintaining a user specified linear set point.
2 Modelization

The purpose of system modelization is to provide an accurate mathematical description of the system. Once this description is obtained, the numerical constants of the system will be substituted into the equations to provide a working representation of the system. From there this model can be simulated and manipulated to obtain gain values on which the state feedback control is based.

This section will derive two sets of mathematical models of the system. The first model is a set of nonlinear differential equations. The second model is a set of linearized differential state equations.

Due to the difficulty level of nonlinear control and the scope of this project, the nonlinear equations will not be used for control calculations. However, the linearized equations provide a good estimate of the system under conditions mentioned in this chapter, and will be used for all control calculations.

2.1 Inverted Pendulum System

The physical model of the inverted pendulum on a moving cart is seen in figure 1. The inverted pendulum system is free to move only in the X-Y plane. The symbolic descriptions are shown in table I.

![Figure 1: Inverted Pendulum System](image-url)
Using the physical model of the inverted pendulum from figure 1, we can now derive the differential equations that describe the system as seen in the following section.

2.2 Nonlinear Dynamical Model

The equations describing the center of gravity of the pendulum rod with respect to the origin on the Cartesian axis of reference can be written as

\[ x_G = x + l \sin \theta \]
\[ y_G = l \cos \theta \]

The balance of forces of the rod in the vertical direction with respect to its center of gravity is given by

\[ m \frac{d^2}{dt^2} (l \cos \theta) = V - mg \]

Completing the differentiation above gives

\[ mg - ml \dot{\theta} \sin \theta - ml \dot{\theta}^2 \cos \theta = V \]  \hspace{1cm} (2-1)

The balance of forces of the rod in the horizontal direction with respect to its center of gravity is given by

\[ m \frac{d^2}{dt^2} (x + l \sin \theta) = H \]

Completing the differentiation above gives
The balance of forces of the cart in the x direction is described by

\[ M \frac{d^2 x}{dt^2} = F - H \]  

Completing the differentiation above gives

\[ M\ddot{x} = F - H \]  

The balance of rotational motion of the pendulum rod around its center of gravity is given by

\[ I\ddot{\theta} = Vl\sin{\theta} - Hl\cos{\theta} \]  

The nonlinear dynamical equations are found by substituting (2-2) into (2-3), and (2-1) into (2-4) as follows

\[ (M + m)\ddot{x} + m l \dot{\theta} \cos{\theta} - ml\dot{\theta}^2 \sin{\theta} = F \]  

\[ (I + ml^2)\ddot{\theta} + m l \ddot{x} \cos{\theta} - mgl\sin{\theta} = 0 \]

Therefore, the nonlinear equations (2-5) and (2-6) mathematically describe the inverted pendulum on a cart.

### 2.3 Linearization of the Dynamical Equations

To be able to apply a linear controller to the inverted pendulum system for stabilization, we need to linearize the dynamical equations (2-5) and (2-6).

Under stabilization conditions of the pendulum we assume the angle of the rod, \( \theta \), is small. By assuming the angle of the rod is always small, the dynamical equations may be linearized by applying the following mathematical rules.

For small values of \( \theta \):

\[ \dot{\theta}^2 \approx 0, \quad \sin{\theta} \approx \theta, \quad \cos{\theta} \approx 1 \]

By applying the above rules for small values of \( \theta \), equations (2-5) and (2-6) transform to

\[ (M + m)\ddot{x} + m l \ddot{\theta} = F \]  

\[ (I + ml^2)\ddot{\theta} + m l \ddot{x} \cos{\theta} - mgl\sin{\theta} = 0 \]  

\[ \dot{\theta}^2 \approx 0, \quad \sin{\theta} \approx \theta, \quad \cos{\theta} \approx 1 \]
Therefore, the linearized equations (2-7) and (2-8) mathematically describe the inverted pendulum on a cart for small values of $\theta$.

### 2.4 Control Force / Electro-Mechanical System Equations

The control force, $F$, of the system is delivered by a DC-motor mounted on the cart. The control input to the motor is measured in voltage, $V$, while the control force delivered is in Newtons. The control force, $F$, must be related to the input voltage, $V$, of the motor attached to the cart. This involves deriving the electro-mechanical equations of the motor and solving these with respect to the control force and the motor voltage.

Assuming negligible armature inductance, the mathematical model of the motor is given by

$$V = R_A I_A + K_m \omega_m$$  \hspace{1cm} (2-9)

where $V$ is the input voltage to the motor, $R_A$ is the armature resistance of the motor, $I_A$ is the armature current drawn by the motor, $K_m$ is the motor torque constant, and $\omega_m$ is the motor speed.

The mechanical equation relating the motor torque to the input armature current is given by

$$\tau_m = K_m K_g I_A$$  \hspace{1cm} (2-10)

where $\tau_m$ is the torque produced at the output shaft of the motor, and $K_g$ is the gear ratio of the gear head on the motor if present.

The linear force applied to the cart by the motor can be described by

$$F = \frac{\tau_m}{r}$$  \hspace{1cm} (2-11)

where $r$ is the radius of the output gear.
The angular velocity of the output shaft relates to the linear velocity of the cart by

\[ \omega_m = \frac{\dot{x}}{r} K_g \]  

(2-12)

where \( \dot{x} \) is the linear velocity of the cart.

We may now combine (2-11) into (2-10) and solve for \( I_A \) obtaining the following

\[ I_A = \frac{Fr}{K_m K_g} \]  

(2-13)

Finally, combining (2-12) and (2-13) into (2-9) we obtain an equation relating the input motor voltage \( V \) to the linear force \( F \) applied to the cart given by

\[ F = \frac{K_m K_g}{r R_A} V - \frac{K_m^2 K_g^2}{r^2 R_A} \dot{x} \]  

(2-14)

2.5 State Space Representation

Now that we have the linearized dynamical equations and the control force equation relating the motor voltage to a linear force, we can substitute (2-14) into (2-7) and rearrange to solve for the linear acceleration \( \ddot{x} \) as follows

\[ \ddot{x} = \frac{K_m K_g}{(M + m) R_A r} V - \frac{K_m^2 K_g^2}{(M + m) R_A r^2} \dot{x} - \frac{ml}{(M + m)} \ddot{\theta} \]  

(2-15)

Similarly, we may rearrange (2-8) to solve for the angular acceleration \( \ddot{\theta} \) as follows

\[ \ddot{\theta} = \frac{mgl}{(I + ml^2)} \theta - \frac{ml}{(I + ml^2)} \ddot{x} \]  

(2-16)

Now we will define the state variables as \( x_1 = \theta \), \( x_2 = \dot{\theta} \), \( x_3 = x \), \( x_4 = \dot{x} \), where \( \theta \) is the angular position of the rod, \( \dot{\theta} \) is the angular acceleration of the rod, \( x \) is the linear position of the cart, and \( \dot{x} \) is the linear acceleration of the cart.

We may now define our state vector as \( X = (\theta, \dot{\theta}, x, \dot{x})^T \). The state space representation is obtained using state equations (2-15) and (2-16) as follows
\[
\dot{X} = AX + BV, \\
y = CX
\]  
(2-17)

where

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{(M + m)gml}{(M + m)I + mml^2} & 0 & 0 & \frac{K_g^2 K_m^2 ml}{r^2 R_A (MI + ml + mml^2)} \\
0 & 0 & 0 & 1 \\
-\frac{gm^2 l^2}{(M + m)I + mml^2} & 0 & 0 & \frac{-l^2 mK_g^2 K_m^2 - IK_g^2 K_m^2}{r^2 R_A (MI + ml + mml^2)}
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0 \\
-\frac{K_g^2 K_m ml}{r R A (MI + ml + mml^2)} \\
0 \\
\frac{ml^2 K_g^2 K_m r + K_g K_m lr}{r^2 R_A (MI + ml + mml^2)}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

3 Control Design

The self-erecting inverted pendulum has a control design for the swing up, and a separate control design for the stabilization. The open loop swing up controller brings the pendulum upright close to the unstable point of equilibrium. Once the angular position has reached a software adjustable capture range, the closed loop stabilization controller takes over.

3.1 Open Loop Swing up Control Design

Swinging the pendulum rod upright using minimal energy is achieved when the oscillating control input frequency is the natural frequency \( \omega_o \) of the rod. To determine the natural frequency of the rod we must first obtain an equation of motion for the pendulum rod as follows

\[
I_o \ddot{\theta} + \frac{1}{2} mgl \sin \theta = 0
\]  
(3-1)
where
\[ I_o = \frac{1}{3} ml^2 \]  
(3-2)

Now, by substituting (3-2) into (3-1) we obtain
\[ \ddot{\theta} + \frac{3g}{2l} \sin \theta = 0 \]  
(3-3)

Comparing (3-3) with the characteristic equation for a second order system, we obtain the natural frequency equation as
\[ \omega_n = \sqrt{\frac{3g}{2l}} \] (rad / s)  
(3-4)

The open loop control function is given by
\[ V = A \sin(\omega_n t) \]  
(3-5)

The control function of (3-5) has two software adjustable values, the natural frequency \( \omega_n \) and the gain \( A \).

The gain \( A \) of the control function adjusts the amplitude of the sine wave function. Since the motor voltage is software limited to \( \pm 7.5V \) during swing up, any value of \( A \) above this value will be saturated.

### 3.2 System Controllability

Before starting the control design it is important to determine the controllability of the system. The system must be controllable if convergence of the state variables is to be achieved.

To determine system controllability we must find the solution to the \((n \times n)\) controllability matrix \( C_o \) given by
\[ C_o = [B \ AB \ A^2B \ A^3B] \]  
(3-6)

The system is controllable if the solution to (3-6) has rank \( n \) where \( n = 4 \). Substituting in the matrices \( A \) and \( B \), and determining the rank of \( C_o \) yields a rank = 4 so it is therefore

10
determined that this system is completely controllable and we may continue with the control design.

3.3 Closed Loop Stabilization Control Design

The stabilization control design is based on linear quadratic regulator (LQR) design with a tracking controller. The objective of this controller is to stabilize the pendulum rod in the upright unstable point of equilibrium while maintaining a software adjustable linear set point position.

The LQR design will effectively return the state feedback gains needed to ensure stability of the system. However, to bring the steady state error of the linear position to zero, a tracking controller is added by integrating the error of the cart position, relative to the linear set point, over time. The gain adjustment of the integration result allows control over the zero steady state error convergence time.

The Matlab block diagram of the system is seen below in figure 2.

![Simulink Control Design Block Diagram](image)

**Figure 2: Simulink Control Design Block Diagram**
From the block diagram it can be seen that the control design uses state feedback. The gain values for the state variables are denoted by $K$, a 1x4 matrix, and are the desired values to achieve stabilization.

The gain block for the integration of the linear position error is denoted by $Ki$ and is also the desired value to achieve zero steady state linear position error.

The saturation block is necessary to represent the experimental system as accurately as possible. Since $V(t)$, the calculated motor voltage into the system, may reach voltages higher than acceptable values for the motor, a saturation function is embedded in the software clipping the motor voltage signal to ±6 Volts.

The control law to be implemented is given by

$$V(t) = -KX(t) + Ki\int_0^t (x_d - x)dt$$

(3-7)

To solve for $Ki$ we must define $\xi$ as the integral of the linear position error as follows

$$\xi = \int_0^t (x_d - x)dt$$

(3-8)

where $x$ is the linear position given by $CX$. Differentiating (3-8) and subbing in $CX$ for $x$ yields the linear position error given by $\dot{\xi}$ as follows

$$\dot{\xi} = x_d - CX$$

(3-9)

The state space representation of the system is now given by

$$\dot{X} = AX + BV,$$

$$\dot{\xi} = x_d - CX$$

$$y = CX$$

(3-10)

Now, we can rewrite (3-10) to include $\dot{\xi}$ as follows

$$\dot{X} = A^*X + B^*V,$$

$$y = CX$$

(3-11)
where
\[ A^* = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad B^* = \begin{bmatrix} B \\ 0 \end{bmatrix} \]

We may now rewrite the control law given in (3-7) as follows
\[ V(t) = -KX(t) + Ki\xi \]

(3-12) may be further reduced to achieve
\[ V(t) = -K^*X^* \]

where
\[ K^* = [K, -Ki], \quad X^* = \begin{bmatrix} X \\ \xi \end{bmatrix} \]

Using the LQR method of design, the state feedback gain matrix \( K^* \) must be determined so the system state variables converge to zero ensuring system stability, and to minimize the performance index \( J \) given by
\[ J = \int_0^\infty (X^TQX + V(t)^TRu)dt \]

(3-14)

The matrix \( Q \) represents the energy in the state variables. The vector \( R \) represents the energy in the control. The two variables may be adjusted to obtain the desired performance of the system.

Once appropriate values of \( Q \) and \( R \) are chosen, matrix \( P \) must be determined by solving the Ricatti equation given by
\[ PA^* + A^*TP - PB^*R^{-1}B^*TP + Q = 0 \]

(3-15)

Now, the state feedback gain matrix \( K^* \) may be determined by
\[ K^* = R^{-1}B^*P \]

(3-16)
4 Experimental Setup

4.1 Mechanical System

The mechanical system for the self-erecting pendulum has 3 major components. These components include the Cart and Gearing System, the Pendulum Rod, and the Track. The general design for the system was inspired and based around the ‘Inverted Pendulum’ by Quansar Consulting. However, many modifications for the mechanical system and position sensor design were necessary to achieve full $360^\circ$ rotational motion.

The cart is made up of a Top Plate, a Front Plate, two Shaft Plates, and two Linear Bearing Supports. The material used is a lightweight aluminum that was machined with precision. The components housed by the cart include 2 linear bearings, 2 standard radial bearings, 1 small gear, 1 large gear, a short shaft, and an electrical terminal block.

This pendulum rod is also made of lightweight aluminum. Because of the nature of this project, it is important that the rod is precisely mounted orthogonal or at a $90^\circ$ angle to the shaft on the cart. It is also important that the integrity of the rod is maintained. If the rod begins to warp it is suggested that this component be replaced. It is important when the rod is in the upright unstable point of equilibrium that this point is physically $0^\circ$.

The track is made up of two end plates, a track-housing guide, a steel guide, and a molded track. The manufacturing of the track is also vital to the project. It is important that the track be built so there is a minimal amount of friction present. If the steel guide is warped in any way, it can cause an unwanted amount of friction to the linear cart motion.

4.2 Motor and Power Module Driver

The motor used for this project is a high torque, high speed, DC, permanent magnet mini motor. Finding the motor for this project proved extremely difficult as the physical size limitations narrowed the choices significantly.
The max speed and the max torque of the motor are very important specifications to consider when designing this system. This project requires a large amount of torque for the swing-up control of the pendulum. High speeds are also necessary for swing-up and to compensate for disturbances once upright.

The power module motor driver contains one ±12 Volt Linear Power supply and an 80W Operational Amplifier. The ±12 Volt power supply drives the op-amp as well as other external devices used by the pendulum system. It is important to consider the maximum current draw of the motor at the maximum terminal voltage being supplied, as this current must be supplied by the op-amp driving the motor.

The basic operation of the power module motor driver for this pendulum system is to provide an inverting unity gain signal of the signal provided by the computer, and allow the motor to draw the current necessary to drive it.

4.3 Position Sensing Electronics

There are two position sensors for this project. The first is a 10-turn potentiometer used to measure the linear position. The second position sensor is a 2 channel optical incremental encoder used to measure the angular position and direction of the pendulum rod.

The linear position sensor is a common 10-turn potentiometer mounted to the cart. There is a gear mounted on the shaft of the potentiometer which combined with the maximum 10-turns translates to a linear movement capable of covering the entire distance of the track. By using a simple voltage divider network, the voltage at the wiper of the potentiometer is connected to an Analog-to-Digital input on a data acquisition board interfaced with the computer. It is calibrated to produce 0 Volts when the cart is at the center position of the track.

The angular position is measured by means of an optical incremental encoder. The encoder used has a resolution of 360 pulses per revolution (PPR). The shaft of the
encoder has been attached to a gear network to increase the resolution of the angular position measurement.

For the encoder used in this project, the output is near sine wave. Ideally a square wave, TTL output would allow interface to the computer much easier. However, a simple comparator circuit can remedy this situation with ease. It should be noted that comparator circuits can be quite noisy, and proper filtering should be implemented to avoid angular position errors.

### 4.4 Computer Interface Hardware

There are two main interface boards used by the pendulum system. The first is a quadrature encoder card used to count the number of pulses from the optical encoder in both directions to supply a correct angular position when required. The quadrature encoder card was manufactured specifically for this project. The second interface board is a simple data acquisition board used for analog-to-digital and digital-to-analog conversions.

The main component of the quadrature encoder card is a 24-Bit quadrature counter. The counter takes the 2 channels from the optical encoder and converts them into directional pulse counting. Therefore, as the pendulum swings clockwise and counterclockwise, the counter will increment and decrement respectively. The 24-Bit counter also increases the resolution of the optical encoder by a factor of 4. This is achieved by registering a pulse for each rising and falling edge of the original input pulses from both channels A and B.

The purpose of the data acquisition board is to convert the linear position sensor voltage into a digital value, and to output a voltage to the power module motor driver from a calculated digital value from the computer. The data acquisition board is capable of sampling at a rate of 100kHz. However, the sampling rate is software limited, which is mentioned in the following chapter.
5 Software Interface

The software created to control the Self Erecting Inverted Pendulum was designed using Visual Basic for the excellent graphical interface and user-friendly design. However, there are limitations with this software. One important limitation is the maximum sampling rate. With enhanced timer controls the maximum sampling rate achievable with this software is 1kHz.

The design of the software allows the user to start the system data acquisition and swing up the pendulum with a command click, or manually bringing the pendulum upright. When the pendulum rod is within a user specified inner capture range the program will begin control calculations and maintain stability of the pendulum rod. Once stability of the pendulum has been achieved, the system will shut down motor output for three possible conditions. The first is if the pendulum rod reaches a plus or minus angular position greater than the user specified outer capture range. The second is if the cart reaches a plus or minus linear position greater than the user specified cart shutdown limit. The third motor shutdown condition occurs when the stop command button is clicked or the program is exited.

Figure 3 depicts the main form of the software interface. This simple design allows for easy input of control parameters, and displays the necessary position information to the user.

The user may also alter the system settings. The system settings form is seen in figure 5. This form includes crucial settings for the system including base addressing, sampling time and inner and outer capture range for the pendulum angle. Other settings include safeties such as a motor shut down upon a critical linear position. This can potentially avoid damage to the linear position sensor and the mechanical system itself.

Other adjustable settings include the swing up frequency and gain.
Figure 3: Software Interface Main Form

Figure 4: Digital Filter Settings Form
The digital filter settings allow the user to control the cutoff frequency of the linear position, angular position and the motor output. Each filter uses a single pole filter programmed into the software.

The filtering of the input measurements from the linear position sensor and the angular position sensor are imperative for accurate readings. Due to the nature of the designed controller, any noise infiltrating the system that is measured can adversely affect the stability of the pendulum rod, and the position of the cart. A single spike of noise voltage induced into the system could be considered a large position change is a very short time. This translates to a very large velocity calculation for either the linear or angular position on the incidental sample, and the control calculation can become erratic and lose stability of the pendulum. A digital filter on the motor voltage output also helps reduce calculation noise, which can also cause erratic motor behavior.
6 Simulations and Experimental Results

6.1 Numerical Control Design Example

The numerical constants of the system are seen in Table II and are necessary for further calculations. The simulation and experimental tests can be analyzed more accurately if the same control design results are applied to both.

<table>
<thead>
<tr>
<th>Symbolic Notation</th>
<th>Description</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>Mass of the cart</td>
<td>0.600</td>
<td>kg</td>
</tr>
<tr>
<td>$m$</td>
<td>Mass of the medium rod</td>
<td>0.228</td>
<td>kg</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational constant</td>
<td>9.81</td>
<td>$m/s^2$</td>
</tr>
<tr>
<td>$l$</td>
<td>Distance to center of the medium rod</td>
<td>0.3225</td>
<td>m</td>
</tr>
<tr>
<td>$L$</td>
<td>Length of medium rod</td>
<td>0.645</td>
<td>m</td>
</tr>
<tr>
<td>$K_g$</td>
<td>Gear ratio</td>
<td>1</td>
<td>$turn(s)$</td>
</tr>
<tr>
<td>$R_A$</td>
<td>Armature resistance</td>
<td>1.4</td>
<td>$\Omega$</td>
</tr>
<tr>
<td>$r$</td>
<td>Radius of the output gear</td>
<td>0.00645</td>
<td>m</td>
</tr>
<tr>
<td>$I$</td>
<td>Moment of inertia of the medium rod</td>
<td>0.007905</td>
<td>$kg\cdot m^2$</td>
</tr>
<tr>
<td>$K_m$</td>
<td>Motor torque constant</td>
<td>0.0102</td>
<td>$N\cdot m/A$</td>
</tr>
<tr>
<td>$\omega_m$</td>
<td>Rated motor speed</td>
<td>3100</td>
<td>$r.p.m.$</td>
</tr>
</tbody>
</table>

Matrices $A$, $B$ and $C$ given in chapter 2.5 are shown numerically as follows

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 28.7512 & 0 & 0 & 6.3228 \\ 0 & 0 & 0 & 1 \\ -2.5532 & 0 & 0 & -2.7189 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -3.9982 \\ 0 \\ 1.7193 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

To begin the controller design we must first obtain matrices $Q$ and $R$. After some experimental simulations the optimal values for $Q$ and $R$ were obtained and are given by
\[ R = \begin{bmatrix} 0.0035 \end{bmatrix}, \quad Q = \begin{bmatrix} 20 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

It can be seen that the energy for the state variables \( \theta \) and \( x \) have been increased to 20 and 30 respectively, from the identity matrix, to obtain a more optimal performance for this system.

Solving the Ricatti equation given by (3-15) yields

\[
P = \begin{bmatrix}
4.2909 & 0.8881 & 3.8657 & 1.9588 & -0.6439 \\
18.5918 & 3.8657 & 25.9091 & 8.7760 & -4.3082 \\
9.4701 & 1.9588 & 8.7760 & 4.4037 & -1.4630 \\
-3.0954 & -0.6439 & -4.3082 & -1.4630 & 6.2142 
\end{bmatrix}
\]

From (3-16) we may determine \( K^* \) as follows

\[
K^* = \begin{bmatrix} -249.7863 & -52.3224 & -105.0393 & -74.4032 & 16.9031 \end{bmatrix}
\]

Therefore the desired state feedback gains and integral gain are given by \( K \) and \( Ki \) respectively as follows

\[
K = \begin{bmatrix} -249.7863 & -52.3224 & -105.0393 & -74.4032 \end{bmatrix}, \quad Ki = \begin{bmatrix} -16.9031 \end{bmatrix}
\]

It is now possible to model the nonlinear system in Matlab and simulate the designed controller as seen in the following section.

### 6.2 Simulation Results

The Matlab simulation will only test the pendulum when in the upright position. However, it will consider initial angular and cart positions as well as disturbances, and allow linear set point adjustments.
When simulating the results of the controller designed it is important to represent the system with the nonlinear equations. This is necessary to obtain results as close as possible to the real system. In chapter 2 the dynamical system equations were linearized only to obtain system equations that would allow a linear control design for the system. However, this control design will be implemented on the physical nonlinear system, and must be simulated as such.

The nonlinear system equations are given by (2-5) and (2-6). We can substitute (2-14) into (2-5) and rearrange to solve for the linear acceleration \( \ddot{x} \) as follows

\[
\ddot{x} = \frac{K_n K_G}{(M + m) r R_A} V - \frac{K_m^2 K_G^2}{(M + m) r^2 R_A} \ddot{x} + \frac{ml}{(M + m)} \dot{\theta}^2 \sin \theta - \frac{ml}{(M + m)} \dot{\theta} \cos \theta \quad (6-1)
\]

Now we can substitute in the constant values to obtain a working numerical equation as follows

\[
\ddot{x} = 1.36421 V - 2.15736 \ddot{x} + 0.0888(\dot{\theta}^2 \sin \theta - \dot{\theta} \cos \theta) \quad (6-2)
\]

Similarly, we may rearrange (2-6) and solve for the angular acceleration \( \ddot{\theta} \) as follows

\[
\ddot{\theta} = \frac{mgl}{(I + ml^2)} \sin \theta - \frac{ml}{(I + ml^2)} \dot{x} \cos \theta \quad (6-3)
\]

Again, by substituting in the constant values we obtain a working numerical equation given by

\[
\ddot{\theta} = 22.81357 \sin \theta - 2.32554 \dot{x} \cos \theta \quad (6-4)
\]

For (6-2) and (6-4) the numerical coefficients will be named for identification as follows. Gain1 = 1.3642, Gain2 = 2.15736, Gain3 = 0.0888, Gain4 = 2.32554, Gain5 = 22.81357.

The nonlinear representation of the system may be seen in figure 6 below.
Now, by using the calculated values for $K$ and $Ki$ we can run a simulation and view the results obtained. The simulation parameters are given below.

- Initial Angular Position: 0.398 radians
- Linear Set Point (initially 0m): 0.2m at time 7.5s
- Disturbance Introduction: 0.2 radians at time 5s
- Disturbance Cancellation: -0.2 radians at time 5.2s
- Simulation Time: 20s

The results obtained from the simulation are seen in figure 7 and figure 8 below. It is apparent that the calculated motor voltage has been limited to a ±5 Volt limit to more accurately represent the physical system. For the physical system the voltage saturation is necessary to avoid drawing too much current during swing up, eliminate possibilities of exceeding maximum terminal voltage and to achieve a better response from the motor.
From the results of the simulation it can be seen that at time 5 seconds, a disturbance to the angular position was introduced and cancelled 0.2 seconds later. To compensate the motor voltage responded accordingly, changing the linear position and subsequently the angular position and maintained stability converging both positions close to zero.

It can also be seen that at time 7.5 seconds, the desired linear position set point changed from 0m to 0.2m and the corresponding linear position gradually began to track to this set point. However, even after a simulation time of 20 seconds the linear position did not quite meet the desired set point. This implies that a larger integral error gain $Ki$ is required to reduce this convergence time.

### 6.3 Experimental Results

To ensure an accurate comparison between the simulated results and the experimental results, the same state feedback gain $K$ and integral gain $Ki$ were used for the physical system tests. These gains may be seen above in section 6.1.

The system parameters are slightly different from the simulation in that the initial angular position is the stable point of equilibrium at 180°. Additionally the swing up control has...
been used to erect the pendulum in the upright position. The calculated swing up variables are shown below

\[ \omega_n = \sqrt{\frac{(3)(9.81)}{(2)(0.645)}} = 4.776 \ (rad / s), \ A = 7.3 \]

The natural frequency of the pendulum rod \( \omega_n \) has been calculated above but does not consider friction in the system. The measured natural frequency has been found as \( \omega_n = 4.25 \ (rad / s) \).

The system gain \( A \) is shown to be greater than the \( \pm 5 \) Volt saturation cutoff limit of the simulation. The software has been programmed with two terminal voltage saturation points. The first is set to \( \pm 7.5 \) V during swing up, and the second is set to \( \pm 5 \) V once the stabilization controller has initiated. This was necessary to achieve a cart driving force great enough to erect the pendulum rod. At the \( \pm 5 \) V limit, the motor was not able to produce enough energy to swing the rod upright. However, using the \( \pm 7.5 \) V limit during stabilization caused the system to be less robust. This is explained in greater detail in the conclusions. In any case, once the pendulum rod has erected, the simulations should accurately represent the physical system.

The experimental results of the physical system can be seen below in figure 9 and figure 10.

Once the swing up controller erected the pendulum, the stabilization controller initiated at roughly 5.5 seconds. After stabilization was achieved a small disturbance was introduced to the pendulum rod at about 10 seconds. This disturbance was introduced by lightly ‘tapping’ the pendulum rod in one direction with a small force from the hand. Again, at roughly 17 seconds the linear set point was changed from 0 cm to 25 cm.
Figure 9: Angular and Linear Position Response of the Physical System

Figure 10: Calculated motor Voltage after Saturation of the Physical System

7 Conclusions

When viewing the results of the experimental tests with the physical system in figure 9 and figure 10 there are a number of interesting observations to be made.

It is important to note the angular position response of the physical system during swing up. It appears that the angle is crossing the 0° threshold on each pass of the cart. This would imply a full revolution of the pendulum rod. However, this is not the case.

When viewing the software interface main form seen in figure 3, one can see the angular position measurement has been set up with 0° at the top, and 180° and -180° both
meeting at the bottom of the circle. Due to this configuration, as the pendulum swings through the bottom of the measurement circle one sample may indicate an angle of near +179.8° while the next sample may indicate an angle of near –179.8°. The system interprets this as a full revolution due to the digital filter on the angular position, which in effect slows the change from one sampled angular position to the next. This phenomenon also has other adverse effects. When the system calculates the angular velocity it is dependant on the angular position measured at two time intervals. When the pendulum rod swings through the bottom of the measurement circle, there is a very large angular velocity calculated for one sample due to this problem. The control of the pendulum for this system is not affected by this error in angular velocity because when the pendulum is swinging up it is not dependent on the angular velocity, angle, linear position or linear velocity due to the open loop nature of the controller.

The calculated motor voltage seen in figure 10 shows a quite noisy response. Even with the digital filter added to the output of this calculation, there is a substantial amount of calculation noise present. However, it is very important to note the peak values at different times during this test. When a disturbance was introduced at 10 seconds, the motor voltage spiked up to roughly 5 V. The other various spikes in the response of the motor voltage after this disturbance do not peak much over 4 V at times. The response of the motor is the important factor for this plot seen. The motor does not begin to generate much driving force until it nears the ±4 V range. For every slight linear position correction needed to maintain a stable system, a motor voltage value is calculated and is send to the system. When the system does not respond with a linear position movement the next sample shows a greater error in the positions, and the next calculated motor voltage is greater. This continues until the position is corrected.

Perhaps the most important change needed for the physical system is the DC permanent magnet mini motor. When ordering this particular motor, the specifications seemed to be sufficient for this system. The necessary torque, speed and terminal voltage along with the physical size were all within the ranges required. However, the overall response of the motor was clearly not considered.
To better this system, a motor with an improved response is necessary to increase the robustness of the system and enhance disturbance recovery and swing up responses. The motor used for the experimental tests does not produce a great enough torque at lower terminal voltages. As the terminal voltage increases, the torque also increases and the motor is able to produce a greater driving force. However, during some experimental testing when the terminal voltage was allowed to increase to values above ±5 V, it was found that the motor was still slow to respond, but when it did respond it created far too much torque at greater speeds and decreased the stability of the system. A motor with a ‘flatter’ response would ideally allow a greater terminal voltage to be applied, and improve the overall system robustness.

After many experimental tests the repeatability of the swing up portion of the system is not great. Since the swing up controller is open loop, any disturbances such as slight bends in the electrical harness attached to the cart create linear friction causing the system to perform differently each time. Improvements could be implemented by designing a closed loop controller for the swing up portion dependant only on the linear position and linear velocity. A trajectory-tracking controller would allow the cart to more accurately follow the sine wave swing up function.
Acknowledgments

The author would like to thank the following individuals for their tremendous support throughout the course of this project. Brett Blyth – Student Project Member, Christiaan Woodfield – Student Project Member, Dr. A. Tayebi – Project Supervisor, Kailash Bhatia – Machinist and mechanical specialist, Warren Paju – PCB Manufacturing and component locating support, Manfred Klein – Component supply and locating support, Bruce Misner – Computer and programming support, Ed Drotar – Mechanical measurement support, Dr. K. Liu – Swing up modelization theory support.

References


