

From Van der Pol to Chua : An Introduction to Nonlinear Dynamics and Chaos for Second Year Undergraduates

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Abstract—This work shows how one can obtain Chua’s circuit with a cubic nonlinearity from the classic Van der Pol oscillator. The approach helps in progressively advancing from the Hopf bifurcation phenomenon in the Van der Pol oscillator to the period-doubling bifurcations in Chua’s circuit. We also place emphasis on mathematical simulation of the dynamic system and physical circuit realization on a breadboard. This systematic methodology has proved invaluable in explaining the phenomenon of nonlinear dynamics and chaos to the curious undergraduate. The student is assumed to have a background in basic differential equations (equilibrium points, stability, linearization) and DC circuit theory. This paper is written with the student in mind. A student should be able to use this paper as a “two week lab manual” in an undergraduate course on nonlinear dynamics and chaos.

I. INTRODUCTION

A challenge faced by most undergraduate students when encountering nonlinear dynamics is the apparent disconnect between the different concepts. Also many of the concepts in nonlinear dynamics seem to be too abstract and mathematical. It is vital early in the study of nonlinear dynamics that the student has a “practical feel” for the concepts along with the mathematical details.

This paper is being used as a two week lab in the introductory nonlinear dynamics and chaos course at the Milwaukee School of Engineering. Our school is on a quarter system and the course is targeted towards second year engineering students. Hence we do not cover topics such as rigorous proofs of chaos via topological horseshoe theory. Rather, the goal of the lab and this paper is that the students grasp basic concepts related to bifurcation phenomenon. In the first week, the student studies Hopf bifurcations. In the second week, the student studies period-doubling bifurcations.

Thus in this paper, we explore a method by which we naturally move from the Hopf bifurcation phenomenon to period-doubling bifurcations. The approach we use is to first study the Van der Pol oscillator [14]. This oscillator is a Liénard system and displays a degenerate Hopf bifurcation at the origin. This system can also be simulated and realized physically. We realize the Van der Pol oscillator by replacing the tunnel diode [14] with an

active cubic nonlinearity based on operational amplifiers and analog multipliers. The purpose of doing so is that this realization can be converted to Chua’s circuit with a cubic nonlinearity [9] by simply adding a resistor and a capacitor. We emphasize that the student first adds the linear resistor and determines a resistance value needed for chaos from the circuit equilibrium points. Next they choose a capacitance value based on Shilkinov eigenvalue conditions [3]. Once the student observes chaos they are encouraged to study period-doubling bifurcations. Hence the student moves from the very first electronic oscillator to a circuit that is the paradigm of chaos. Although the first offering of this course had only six students, the response was all positive. A particular response was “Through nonlinear dynamics, our basic circuit concepts were further reinforced and also extended. For example, we got to use analog multipliers and op-amps in positive feedback (current inverters). We are not exposed to analog multipliers, even positive feedback op-amp circuits are limited to Schmitt Triggers.”

The organization of this paper naturally follows our two week lab approach. We first derive circuit equations for the Van der Pol oscillator. This is followed by a stability analysis to mathematically confirm the existence of the Hopf bifurcation. Next we simulate the Van der Pol differential equations. Note that one can use any simulation package, in this paper we use MATLAB [11]. We “close-the-loop” by realizing the Van der Pol oscillator on a breadboard. We follow suite with Chua’s circuit. The paper concludes with a discussion of future work.

II. VAN DER POL OSCILLATOR

A. Circuit Equations

The Van der Pol equation can be represented using the circuit [5] shown in Fig. 1. We will now derive the circuit equations and prove that they correspond to the classic Van der Pol oscillator.

Using Kirchhoff’s current law [8] at v_{C1} in Fig. 1 gives Eq.(1).

$$i_L + i_{C1} + i_R = 0 \quad (1)$$

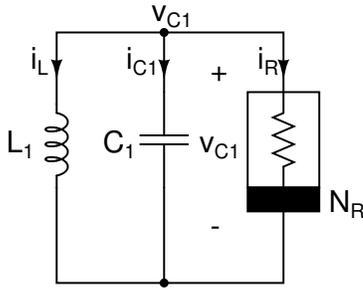


Fig. 1: A circuit model of the Van der Pol oscillator.

Taking the time-derivative of Eq.(1) yields Eq.(2).

$$\frac{di_L}{dt} + \frac{di_{C1}}{dt} + \frac{di_R}{dt} = 0 \quad (2)$$

Using element laws, passive sign convention and the chain rule [8], we rewrite Eq.(2) resulting in Eq.(3).

$$\frac{v_{C1}}{L_1} + C_1 \frac{d^2 v_{C1}}{dt^2} + \frac{di_R}{dv_{C1}} \frac{dv_{C1}}{dt} = 0 \quad (3)$$

In Eq.(3), $i_R(v_{C1})$ is the driving-point characteristic of the nonlinear resistor N_R in Fig. 1. A classic implementation of the Van der Pol oscillator uses a tunnel-diode in series with a voltage source [14] to represent the nonlinear resistor. That is we have an active cubic nonlinearity as shown in Eq.(4).

$$i_R(v_{C1}) = av_{C1} + bv_{C1}^3 \quad (4)$$

We will set $a = -1, b = \frac{1}{3}$ since they are "standard" values for the Van der Pol oscillator [5]. However we will design N_R using the the model [9] in Fig. 2.

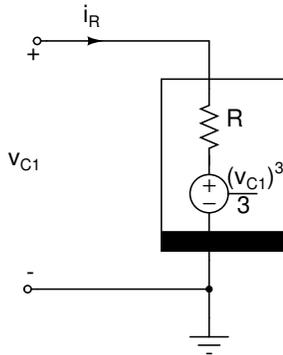


Fig. 2: Circuit model of the nonlinear resistor for the Van der Pol oscillator.

Using Kirchhoff's voltage law [8] in Fig. 2 we have Eq.(5).

$$i_R(v_{C1}) = \left(v_{C1} - \frac{(v_{C1})^3}{3} \right) \frac{1}{R} \quad (5)$$

If $R < 0$, we get Eq.(6).

$$i_R(v_{C1}) = \left(-v_{C1} + \frac{(v_{C1})^3}{3} \right) \frac{1}{|R|} \quad (6)$$

Physically implementing the nonlinear resistor can be done using operational amplifiers and analog multipliers [9] as shown in Fig. 3. Eq.(7) shows the final expression for $i_R(v_{C1})$. The derivation is a good exercise for the student.

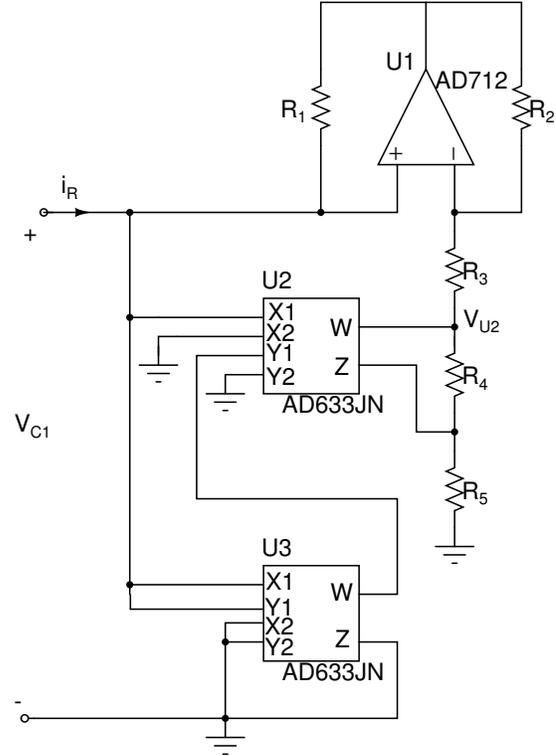


Fig. 3: Circuit implementation of nonlinear resistor. All power supplies are $\pm 15 V$. Op-amp U_1 acts as a current inverter when $R_1 = R_2$, U_2 and U_3 are analog multipliers.

$$i_R(v_{C1}) = \left[-v_{C1} + \left(1 + \frac{R_5}{R_4} \right) \frac{(v_{C1})^3}{100} \right] \frac{1}{R_3} \quad (7)$$

Comparing Eq.(7) to Eq.(6), we see that:

$$\left(1 + \frac{R_5}{R_4} \right) \frac{1}{100} = \frac{1}{3}, R_3 = |R| = R \quad (R > 0) \quad (8)$$

We will choose $R_1 = R_2 = 2.2 \text{ k}\Omega$, $R_4 = 1.1 \text{ k}\Omega$ and $R_5 = 36 \text{ k}\Omega$. The strategy we employ is to first pick R_1, R_2 according to [9]. Then we pick R_4 close to $1 \text{ k}\Omega$. We then solve for R_5 by comparing Eq.(6) and Eq.(7). Substituting Eq.(7) with these resistance values into Eq.(3) and simplifying we get Eq.(9).

$$C_1 \frac{d^2 v_{C1}}{dt^2} + (-1 + v_{C1}^2) \frac{1}{R} \frac{dv_{C1}}{dt} + \frac{v_{C1}}{L_1} = 0 \quad (9)$$

Time-scaling Eq.(9) $\tau = \frac{t}{\sqrt{L_1 C_1}}$ and simplifying, we get Eq.(10) where we have used the dot notation to indicate derivative with respect to τ .

$$\ddot{v}_{C1} - \epsilon (1 - v_{C1}^2) \dot{v}_{C1} + v_{C1} = 0 \quad (10)$$

In Eq.(10), $\epsilon = \sqrt{\frac{L_1}{C_1}} \frac{1}{R}$. We want to emphasize that Eq.(10) is not dimensionless. We have simply time-scaled the equation to obtain the same form as the Van der Pol oscillator. In other words, the current is in milliamperes and the voltage will be on the order of volts. We do not require the dimensionless version because our goal is to synthesize Chua's circuit from Fig. 1, as we will demonstrate later in the paper. But first, we will mathematically confirm the existence of a Hopf bifurcation in the Van der Pol oscillator.

B. Stability Analysis

Using $x = v_{C1}, y = \dot{v}_{C1}$ we obtain Eq.(11), the classic state-space form of the Van der Pol oscillator.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \epsilon \cdot y(1 - x^2) - x \end{aligned} \quad (11)$$

In order to determine the Hopf bifurcation in the Van der Pol oscillator, we first compute the fixed points of Eq.(11).

$$\begin{aligned} 0 &= y^* \\ 0 &= \epsilon \cdot y^*(1 - (x^*)^2) - x^* \end{aligned} \quad (12)$$

Hence the fixed points of the system are $(x^*, y^*) = (0, 0)$. Evaluating the Jacobian of Eq.(11) at the equilibrium points we get Eq.(13).

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix} \quad (13)$$

The eigenvalues of the matrix in Eq.(13) are purely imaginary when $\epsilon = 0$. If $\epsilon > 0$, then we have complex conjugate eigenvalues with positive real parts. If $\epsilon < 0$, then we have complex conjugate eigenvalues with negative real parts. In other words, by definition [6], a Hopf bifurcation occurs at $\epsilon = 0$.

Now that we have confirmed the existence of a Hopf bifurcation, we will reproduce the Hopf bifurcation via mathematical simulations and a physical circuit realization. We will first finish choosing parameters for the Van der Pol oscillator. We first fix $L_1 = 18 \text{ mH}$ and $C_1 = 6.8 \text{ nF}$. Note that these are robust parameter values for producing chaos in Chua's circuit from [9]. If we use $\epsilon = 1.02$, we determine $R = 1.6 \text{ k}\Omega$. The justification for choosing an ϵ closer to 1 is that a very large value of ϵ implies a very strong nonlinearity in Eq.(10). This may lead to a large amplitude limit cycle in Chua's circuit. A similar argument applies to a very small value of ϵ . We educate the student on these subtle concepts once they finish exploring Chua's circuit.

C. Results from System Simulation and Physical Realization

Fig 4 show our phase-portrait results from simulations and physical realization. We imported all data into MATLAB for ease of plotting.

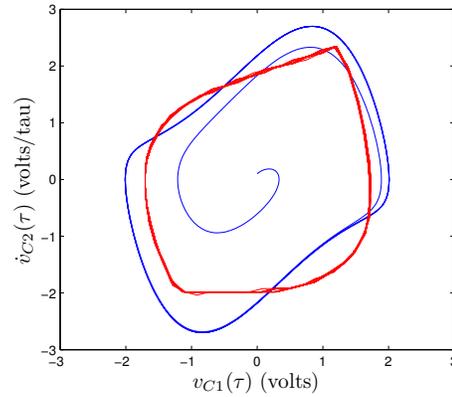


Fig. 4: A superimposed phase plot of results from MATLAB simulation (blue) and physical circuit realization (red). Note the y-axis time scale is dimensionless. ode45 integration routine was used in MATLAB with relative and absolute tolerances of $1 \cdot 10^{-7}$. Initial conditions are $(0,1,0)$. In order to experimentally measure \dot{v}_{C1} , we used a current probe to measure i_{C1} and scaled using C_1 . We then imported the data into MATLAB and time-scaled to τ . You may also need to compensate for probe impedance.

III. CHUA'S CIRCUIT

The students are now asked to obtain chaos from Fig. 1 by adding a resistor and a capacitor. These two elements provide the third state necessary for a continuous time autonomous dynamical system to become chaotic [3].

A. The Circuit Model

The result of inserting resistor R and capacitor C_2 are shown in Fig. 5.

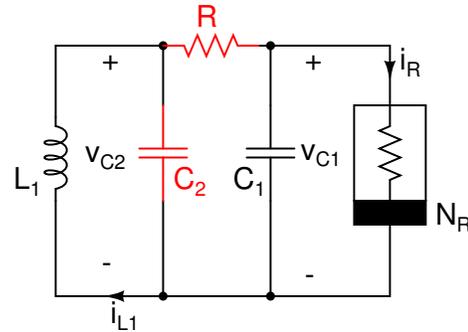


Fig. 5: Circuit model for Chua's circuit. The only elements added to the Van der Pol oscillator are highlighted.

The differential equations in Eq.(14) describe Chua's

circuit in Fig. 5.

$$\begin{aligned} \frac{dv_{C1}}{dt} &= \frac{1}{C_1} \left(\frac{v_{C2} - v_{C1}}{R} - i_R(v_{C1}) \right) \\ \frac{dv_{C2}}{dt} &= \frac{1}{C_2} \left(\frac{v_{C1} - v_{C2}}{R} + i_{L1} \right) \\ \frac{di_{L1}}{dt} &= -\frac{v_{C2}}{L_1} \end{aligned} \quad (14)$$

B. Stability Analysis

First we compute the equilibrium points $(v_{C1}^*, v_{C2}^*, i_{L1}^*)$ in Eq.(14) as shown in Eq.(15).

$$\begin{aligned} 0 &= \frac{1}{C_1} \left(\frac{v_{C2}^* - v_{C1}^*}{R} - i_R(v_{C1}^*) \right) \\ 0 &= \frac{1}{C_2} \left(\frac{v_{C1}^* - v_{C2}^*}{R} + i_{L1}^* \right) \\ 0 &= -\frac{v_{C2}^*}{L_1} \end{aligned} \quad (15)$$

Hence one of the equilibrium points is the origin. Resistance R is chosen such that the other two equilibrium points lie in the negative resistance region of the nonlinear resistor N_R [3]. A choice of $R = 1.9 \text{ k}\Omega$ gives rise to the equilibrium points $(-0.68 \text{ V}, 0, -0.36 \text{ mA})$, $(0.68 \text{ V}, 0, 0.36 \text{ mA})$.

Next we compute the Jacobian of Eq.(14) at the three equilibrium points and impose the first Shilnikov condition [3]. This will lead to a multitude of capacitance values. We picked $C_2 = 68 \text{ nF}$, very close to the value of 70 nF from [9].

C. Results from System Simulation and Physical Realization

Fig 6 shows our phase-portrait results from simulations and physical realization.

IV. CONCLUSIONS AND FUTURE WORK

In this paper we have shown how to systematically move from the Hopf bifurcation in the Van der Pol oscillator to the period-doubling bifurcations in Chua's circuit. The central concept is that a Van der Pol oscillator can use the same cubic nonlinearity as Chua's circuit. Only two parameters need to be tuned when deriving Chua's circuit from our Van der Pol oscillator realization.

As a natural followup, a student wonders about a rigorous proof of chaos in Chua's circuit. Note that we only imposed the first Shilnikov condition. A rigorous proof would require confirming the existence of a homoclinic orbit [3]. Since we have a smooth nonlinear function, the instructor is encouraged to pursue the method of undetermined coefficients [10] to find the homoclinic orbit. Another approach would be for the instructor to start a rigorous analysis of chaos via the bifurcation methodologies presented in [7]. One could also discuss minimizing the number of elements required for chaos [1] in Fig. 5.

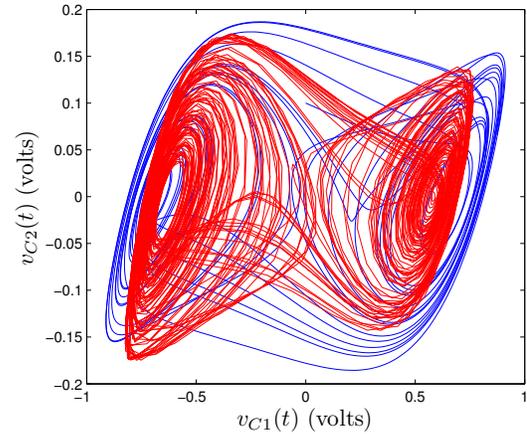


Fig. 6: A superimposed phase plot of results from MATLAB simulation (blue) and physical circuit realization (red). MATLAB initial conditions are $(0,0.1,0)$. ode45 with the same settings from the Van der Pol oscillator were used. Data was sampled using NI-myDAQ [12]. Both x and y data from the oscilloscope were scaled by a factor of 2 to match MATLAB axis scales. We were unable to avoid aliasing in the physical circuit realization (red) data.

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